Regular Paper

メンバシップ関数集合の定義といくつかのメンバシップ関数の諸性質

On the Formalizations of Definition for Set of Membership Functions and Several Types of Membership Functions

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Abstract

In this study, valious sets of membership functions (fuzzy sets) are formalized. At first, the set of membership functions [1,2] is defined. Next, sine curve, cosine curve and Gaussian function that renges are limited to interval $[0, 1]$ are formalized. On the other, not only curve membership functions mentioned above but also membership functions composed of straight lines like triangular and trapezoidal are formalized. Moreover different from the definition in [3] fomalizations of triangular and trapezoidal function composed of two straight lines, minimum function and maxmum functions are proposed.

1 はじめに

ファジィ推論における IF-THEN ルールは複数のファジィ集合（メンバシップ関数）によって構成されている．それゆえ IF-THEN ルールはメンバシップ関数の対とみなされてよい．ファジィ推論やファジィ制御の結果の適否は IF-THEN ルールの設定によるところが大きく，構成するメンバシップ関数が適切に選ばれることによって，ファジィ推論や制御は良好な結果を得る．筆者らはファジィ制御の最適制御に関する研究を行っており，最

適制御問題を汎関数の最小化問題に帰着させて，最適制御を与える IF-THEN ルールの存在性を証明した [4].

本アーティクルはそれらの成果を MIZAR システムを使って検証するために，メンバシップ関数（ファジィ集合）の集合を定義した．その他，三角関数やグラフが折れ線になる区分線形関数を用いたメンバシップ関数の諸性質の定式化を行った．

2　MIZAR におけるメンバシップ関数とファジィ集合

MIZAR においてファジィ理論は筆者らによって [1] により形式化が始まった．そこで，メンバシップ関数は以下の [FUZZY_1 mode] のように値域が $[0,1]$ である関数として定義された．

Listing 1. FUZZY_1 mode

```
definition
 let C be non empty set ;
 mode Membership_Func of C is [.0,1.] −valued Function of C,REAL;
end;
```

当初，ファジィ集合は同論文 [1] 内で集合の要素（変数）とそのメンバシップ関数値との対として次のように定義された．

Let C be a non empty set and let h be a membership function of C.

A set is called a FuzzySet of C, h if :

(Def.2)　It = [: $C, h^\circ C$:].

この定義を用いたファジィ集合論が収録された FUZZY_1，FUZZY_2 および FUZZY_4 の一部の定義，定理は変更され，アーティクル FUZZY_3 は現在削除され欠番となっている．FUZZY_3 が削除された後に，以下の [FUZNUM_1 mode] にてメンバシップ関数（Membership _Func）とファジィ集合（FuzzySet）は同一視することとされた．

Listing 2. FUZNUM_1 mode

```
definition
 let C be non empty set ;
 mode FuzzySet of C is Membership_Func of C;
end;
```

3　メンバシップ関数の形式化

3.1　メンバシップ関数（ファジィ集合）の集合の定義

本アーティクルではメンバシップ関数の集合を以下のように定義した．これにより，ファジィ集合族 [FUZZY_5:2] が定式化された．

Listing 3. FUZZY_5:def 1, FUZZY_5:2

```
definition
 let X be non empty set;
 func Membership_Funcs (X) −> set means :: FUZZY_5:def 1
 for f being object holds f in it iff f is Membership_Func of X ;
end;

theorem :: FUZZY_5:2
Membership_Funcs (REAL) ={f where f is Function of REAL,REAL : f is FuzzySet of REAL};
```

3.2 種々のメンバシップ関数集合

ファジィ理論の実用化に際し，様々なメンバシップ関数が用いられる．その中には，三角型形や台形型，S 型，Z 型のように直線のみで構成された関数や，釣鐘型のような三角関数やガウス関数の値域を $[0, 1]$ に限定した関数がある．以下の定理では，それらの集合が [FUZZY_5:def 1] で定義した Membership_Funcs の部分集合になっていることを示す．

Listing 4. FUZZY_5:15,35,57,61

```
theorem :: FUZZY_5:15
for g be Function of REAL,REAL holds
{f where f is Function of REAL,REAL : for x be Real holds f.x= min(1,max(0, g.x))} c= Membership_Funcs (REAL);

theorem :: FUZZY_5:35
{f where f is Function of REAL,REAL, a,b,c,d is Real: for x be Real holds f.x= max(0,min(1, c∗sin(a∗x+b)+d))}
c= Membership_Funcs (REAL);

theorem :: FUZZY_5:57
{f where f is Function of REAL,REAL, a,b,c is Real: b <> 0 & for x be Real holds f.x= max(0,min(1, exp_R(−(x−a)
      ^2/(2∗b^2))+c))}
c= Membership_Funcs (REAL);

theorem :: FUZZY_5:61
{f where f is Function of REAL,REAL, a,b is Real: for x be Real holds f.x= max(0,min(1, AffineMap (a,b).x))}
c= Membership_Funcs (REAL);
```

3.3 メンバシップ関数のリプシッツ連続性

IF-THEN ルールを構成するファジィ集合（前件部）のメンバシップ関数のリプシッツ連続性は，ファジィ制御における状態方程式の解の存在性のための必要条件の一つとなる [4]．そこで種々のメンバシップ関数のリプシッツ連続性を証明した．以下の定理は任意の関数に対して頭切り計算のリプシッツ連続性を導く．また $c = 1, d = 0$ とすれば任意の関数の値域を $[0, 1]$ に変換が可能で，その関数のリプシッツ連続性の証明に適用しうる．

Listing 5. FUZZY_5:13,69

```
theorem :: FUZZY_5:13
for a,b,c,d being Real holds |. max(c,min(d,a)) − max(c,min(d,b)) .| <= |. a−b .|;

theorem :: FUZZY_5:69
for f be Function of REAL,REAL, a,b,c be Real st
( b <> 0 & for x be Real holds f.x= max(0,min(1, c∗(1−|.(x−a)/b.|))) ) holds f is Lipschitzian;
```

以下の任意の実数 x に対する不等式 $|\sin x| \leq |x|$（[FUZZY_5:40]）は，Lagrange Theorem と三角関数の導関数 [ROLLE:4, SIN_COS:64] より導かれた [5, 6].

Listing 6. ROLLE:4, SIN_COS:64

```
theorem :: ROLLE:4
for x, t being Real st 0 < t holds
for f being PartFunc of REAL,REAL st [.x,(x + t).] c= dom f & f | [.x,(x + t).] is continuous &
f is_differentiable_on ].x,(x + t).[ holds
ex s being Real st ( 0 < s & s < 1 & f . (x + t) = (f . x) + (t * (diff (f,(x + (s * t))))) ) ;

theorem Th63: :: SIN_COS:64
for p being Real holds
( sin is_differentiable_in p & diff (sin,p) = cos . p );
```

Listing 7. FUZZY_5:40

```
theorem :: FUZZY_5:40
for x being Real holds |. sin x .| <= |.x.|;
```

これにより，三角関数を用いたメンバシップ関数のリプシッツ連続性が示せた.

Listing 8. FUZZY_5:45

```
theorem :: FUZZY_5:45
for f be Function of REAL,REAL, a,b,c,d be Real st
for x be Real holds f.x= max(0,min(1, c*sin(a*x+b)+d)) holds f is Lipschitzian;
```

3.4 周期関数型メンバシップ関数

ファジィ推論やファジィ制御の対象事項として，方向や時間，色彩等は十分該当し得る．これらの事項を表すメンバシップ関数（ファジィ集合）は周期性を有する [7]．そこで，それらの推論・制御手法の MIZAR による数学的検証の道具となるべく，周期関数型メンバシップ関数を正弦関数によって定式化した．以下の定理は正弦関数によるメンバシップ関数が周期関数型メンバシップ関数であることを示し，特に [FUZZY_5:28] はその周期が $\frac{2\pi i}{a}$ であることを表している.

Listing 9. FUZZY_5:28,29

```
theorem :: FUZZY_5:28
for F being Function of REAL,REAL, a,b,c,d being Real, i be Integer st
a<>0 & i<>0 & for x be Real holds F.x= max(0,min(1, c*sin(a*x+b)+d)) holds F is (2 * PI)/a * i -periodic;

theorem :: FUZZY_5:29
for F being Function of REAL,REAL, a,b,c,d being Real st
for x be Real holds F.x= max(0,min(1, c*sin(a*x+b)+d)) holds F is periodic;
```

本アーティクルでは正弦についてのみであるが，余弦についても同様の定理が得られる．これについては今後必要に応じて形式化していく.

3.5 三角形型，台形型メンバシップ関数

三角形型および台形型メンバシップ関数は [3] にて以下のように定義されている．三角形および台形の頂点の座標，つまり適合度が 0 や 1 の定義域の値によって定義している．以下，$\mathit{AffineMap}(a, b) = ax + b$ である．

Listing 10. FUZNUM_1:def 7,8

```
definition
 let a, b, c be Real;
 assume that Z1: a < b and Z2: b < c ;
 func TriangularFS (a,b,c) −> FuzzySet of REAL equals :TrDef: :: FUZNUM_1:def 7
 (((AffineMap (0,0)) | (REAL \ ].a,c.[)) +* ((AffineMap ((1 / (b − a)),(− (a / (b − a))))) | [.a,b.])) +* ((AffineMap ((− (1
      / (c − b))),(c / (c − b)))) | [.b,c.]);
end;

definition
 let a, b, c, d be Real;
 assume that Z1: a < b and Z2: b < c and Z3: c < d ;
 func TrapezoidalFS (a,b,c,d) −> FuzzySet of REAL equals :TPDef: :: FUZNUM_1:def 8
((((AffineMap (0,0)) | (REAL \ ].a,d.[)) +* ((AffineMap ((1 / (b − a)),(− (a / (b − a))))) | [.a,b.])) +* ((AffineMap (0,1)) |
      [.b,c.])) +* ((AffineMap ((− (1 / (d − c))),(d / (d − c)))) | [.c,d.]);
end;
```

一方，本アーティクルでは上述のファジィ集合を傾きの異なる 2 本の直線の方程式と MIN，MAX 演算で構成している．つまり，上記の直線の傾き a および切片 b によって定義をした．両者の等しさは下記の定理にて示されている．

Listing 11. FUZZY_5:80,86

```
theorem :: FUZZY_5:80
for a,b,p,q be Real st a > 0 & p > 0 & (1−b)/a < (1−q)/(−p) holds
for x be Real holds (TrapezoidalFS ((−b)/a,(1−b)/a,(1−q)/(−p),q/p)).x
= max(0,min(1, ( (AffineMap (a,b) |].−infty,(q−b)/(a+p).[) +* (AffineMap (−p,q)|[.(q−b)/(a+p),+infty.[) ) .x ));

theorem :: FUZZY_5:86
for a, b, c being Real st a < b & b < c holds
for x being Real holds TriangularFS (a,b,c).x =
max(0,min(1, ( (AffineMap ( 1/(b−a),− a/(b−a) )|].−infty,b.[) +* (AffineMap ( − 1/(c−b),c/(c−b) )|[.b,+infty.[) ).x ));
```

ファジィ推論において，各ルールの推論結果のメンバシップ関数を MAX 演算や和算を使って合成する計算をする際，[FUZZY_5:80] の表記のほうが扱いやすいと考える．

以下の定理は三角型および台形型メンバシップ関数のリプシッツ連続性を示している．

Listing 12. FUZZY_5:88

```
theorem :: FUZZY_5:88
for a,b,p,q be Real, f be Function of REAL,REAL st
a > 0 & p > 0 &
for x be Real holds
f.x = max(0,min(1, ( ((AffineMap (a,b))|(].−infty,(q−b)/(a+p).[)) +* ((AffineMap (−p,q))|([.(q−b)/(a+p),+infty.[)) ) .x ))
holds f is Lipschitzian;
```

この定理では，最小値 0 以上，最大値 1 以下と定めているが，別の定理にて任意の実数で成立していることを示している．さらにこの定理により，グラフが折れ線となっている区分線形関数を用いたメンバシップ関数のリプシッツ連続性が得られた．このことは，

メンバシップ関数を MAX 演算や和算を使って合成した後の関数のリプシッツ連続性を与えるものである．

4 まとめ

メンバシップ関数の集合を定義し，実用にて頻繁に使用されるメンバシップ関数の定式化を行った．また，三角形型，台形型のメンバシップ関数を 2 直線の傾きと切片，MIN，MAX 演算を用いて定式化した．これによって，重心計算などの演算の際に区分線形関数の係数が煩雑になるのを回避できる．

今後はメンバシップ関数集合のコンパクト性とその上に定義された推論計算の汎関数としての連続性に関するアーティクルをそれぞれ作成し，ファジィ最適制御の存在性の検証を行う．

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Mizar article information

Works in Progress

FUZZY_5 Set of Membership Functions
by Takashi Mitsuishi

Summary: We have been working on the formalization of the fuzzy logic. In [1], we defined membership function and encoded some theorems concerning the membership functions. In this article, we present formalization of the set of membership functions. First, we define the set of membership function and show the membership functions whose graphs are various shapes for example, sine, cosine and piecewise linear function. Next, we describe continuity and periodicity of membership function. Moreover different from the definition in [3] fomalizations of triangular and trapezoidal function composed of two straight lines, minimum function and maxmum functions are proposed.

Listing 13. FUZZY_5 - abstract

```
environ

 vocabularies NUMBERS, XBOOLE_0, SUBSET_1, XXREAL_1, CARD_1, RELAT_1, TARSKI,
      FUNCT_1, XXREAL_0, PARTFUN1, ARYTM_1, ARYTM_3, COMPLEX1, FUZZY_1,
      MSALIMIT, FUZNUM_1, REAL_1, ORDINAL2, FCONT_1,SQUARE_1,FUNCT_3,
      RCOMP_1, NUMPOLY1, JGRAPH_2, FUNCT_4,FUNCT_7, SIN_COS,POWER, FUNCT_9,
      INT_1,VALUED_1, FDIFF_1, SIN_COS9, FUZZY_5;
 notations TARSKI, XBOOLE_0, SUBSET_1, RELSET_1, NUMBERS,COMPLEX1, RELAT_1,
      SQUARE_1, STRUCT_0, NORMSP_1, METRIC_1, FUNCT_1, ORDINAL1, INT_1,
      XCMPLX_0,XXREAL_0,XREAL_0, XXREAL_2, VALUED_1,PARTFUN1, FUNCT_2,
      FUNCT_3, FUNCT_4,FUNCT_9, FDIFF_1, MEMBERED, FINSEQ_1, FINSEQ_2,
      SIN_COS, RCOMP_1, MEASURE5, RFUNCT_1, PRE_TOPC, COMPTS_1, TOPMETR,
      TAYLOR_1, POWER, SIN_COS9, FCONT_1, FUZZY_1,FUZNUM_1,LFUZZY_1;
 constructors ZFMISC_1, SUBSET_1, FUNCT_1, XCMPLX_0, XXREAL_0, XREAL_0,
      COMPLEX1, RFUNCT_1, INTEGRA1, SEQ_4, RELSET_1, FUZZY_1, FCONT_1,
      FUNCT_4,FUZNUM_1,NUMPOLY1,LFUZZY_1,RELAT_1,SIN_COS,XXREAL_3,NUMBERS,
      SIN_COS6,TOPMETR,SQUARE_1,TAYLOR_1,POWER, COMPTS_1, RCOMP_1, FINSEQ_1,
      FINSEQ_2, INT_1,ORDINAL1,FUNCT_9, FDIFF_1, SIN_COS9,STRUCT_0, NORMSP_1,
      METRIC_1, VALUED_1, PARTFUN1;
 registrations RELSET_1, NUMBERS, XXREAL_0, MEMBERED, XBOOLE_0, VALUED_0,
      VALUED_1, FUNCT_2, XREAL_0, ORDINAL1, FCONT_1, RELAT_1, TOPREALB,
      FUNCT_4, FUNCT_1, XCMPLX_0, NAT_1, RCOMP_1,FUZNUM_1,NUMPOLY1,
      LFUZZY_1,SIN_COS,XXREAL_3, SIN_COS6, TOPMETR, SQUARE_1, SIN_COS3,
      CARD_3, INT_1, SUBSET_1, SIN_COS9, NORMSP_1,PARTFUN1;
 requirements NUMERALS, REAL, SUBSET, BOOLE, ARITHM ;
 definitions XBOOLE_0, TARSKI, XXREAL_2, FUZNUM_1, NUMPOLY1, FUZZY_1,
      RELAT_1, XXREAL_3, NUMBERS, SIN_COS6, TOPMETR, SQUARE_1, COMPTS_1,
      RCOMP_1, STRUCT_0, INT_1, SUBSET_1, SIN_COS9, VALUED_1, XXREAL_0,
      SIN_COS, FCONT_1,FUNCT_3,PARTFUN1;
 equalities FUZZY_1, RCOMP_1,FUZNUM_1,NUMPOLY1,RELAT_1,XXREAL_3,
      NUMBERS,SIN_COS6,SQUARE_1,FUNCT_4;
 expansions TARSKI, FCONT_1,FUZNUM_1,NUMPOLY1,FUZZY_1,RELAT_1,XXREAL_3,
      NUMBERS,SQUARE_1,FUNCT_4;
 theorems TARSKI, FUNCT_1, ABSVALUE, XBOOLE_0,POWER,TAYLOR_1,SIN_COS,
      COMPLEX1, XXREAL_1,XREAL_0,FCONT_1, XCMPLX_1, RCOMP_1, FUZNUM_1,
      FUNCT_9, SIN_COS4,ROLLE,FDIFF_1,VALUED_1,SIN_COS3, SQUARE_1,
      XXREAL_0, FUNCT_4, XREAL_1, FUNCT_2, FUNCT_3, SIN_COS6,FUZZY_1;
 schemes FUNCT_2;

begin
:::: set of membership functions

definition
let X be non empty set ;
```

```
func Membership_Funcs (X) -> set means
:: FUZZY_5:def 1
for f being object holds
 f in it iff f is Membership_Func of X ;
end;

theorem :: FUZZY_5:1
for X be non empty set, x be object
st x in Membership_Funcs (X)
holds
ex f be Membership_Func of X st x = f & dom f = X;

theorem :: FUZZY_5:2
Membership_Funcs (REAL)
={f where f is Function of REAL,REAL : f is FuzzySet of REAL};


:::: characteristic function (indicator function)

theorem :: FUZZY_5:3
for A,X be non empty set holds
{chi (A,X)} c= Membership_Funcs (X);

theorem :: FUZZY_5:4
{chi([.a,b.],REAL) where a,b is Real : a <= b}
c= Membership_Funcs (REAL);

theorem :: FUZZY_5:5
{chi(A,REAL) where A is Subset of REAL : A c= REAL}
c= Membership_Funcs (REAL);

theorem :: FUZZY_5:6
{f where f is FuzzySet of REAL : ex A be Subset of REAL st f=chi(A,REAL) }
c= Membership_Funcs (REAL);


:::: membership functions using min (max) operation

theorem :: FUZZY_5:7
for f,g be Function of REAL,REAL, a being Real st
g is continuous & for x be Real holds f.x= min(a, g.x)
holds f is continuous;

theorem :: FUZZY_5:8
for F,f,g be Function of REAL,REAL st
f is continuous & g is continuous &
for x be Real holds F.x= min(f.x, g.x)
holds
F is continuous;

theorem :: FUZZY_5:9
for F,f,g be Function of REAL,REAL st
f is continuous & g is continuous &
for x be Real holds F.x= max(f.x, g.x)
holds
F is continuous;

theorem :: FUZZY_5:10
for f,g be Function of REAL,REAL,
a being Real st
g is continuous & for x be Real holds f.x= max(a, g.x)
holds f is continuous;

theorem :: FUZZY_5:11
for f,g be Function of REAL,REAL, a,b being Real st
g is continuous & for x be Real holds f.x= max(a,min(b, g.x))
holds f is continuous;

theorem :: FUZZY_5:12
```

```
for f,g be Function of REAL,REAL st
g is continuous & for x be Real holds f.x= max(0,min(1, g.x))
holds f is continuous;

theorem :: FUZZY_5:13
for a,b,c,d being Real holds
 |. max(c,min(d,a)) − max(c,min(d,b)) .| <= |. a−b .|;

theorem :: FUZZY_5:14:: LeMM01Lip:
for r,s be Real, f be Function of REAL,REAL st
for x be Real holds f.x= max(r,min(s,x))
holds
f is Lipschitzian;

theorem :: FUZZY_5:15
for g be Function of REAL,REAL holds
{f where f is Function of REAL,REAL :
for x be Real holds f.x= min(1,max(0, g.x))}
 c= Membership_Funcs (REAL);

theorem :: FUZZY_5:16
{f where f,g is Function of REAL,REAL :
for x be Real holds f.x= max(0,min(1, g.x))}
 c= Membership_Funcs (REAL);

theorem :: FUZZY_5:17
for f,g be Function of REAL,REAL st
(for x be Real holds f.x= max(0,min(1, g.x)))
holds
f is FuzzySet of REAL;

theorem :: FUZZY_5:18
for f,g be Function of REAL,REAL st
(for x be Real holds f.x= min(1,max(0, g.x)))
holds f is FuzzySet of REAL;


:::: fuzzy Set from trigonometric function

theorem :: FUZZY_5:19
{f where f is Function of REAL,REAL :
ex a,b be Real st for th be Real holds f.th= 1/2∗sin(a∗th+b)+1/2}
 c= Membership_Funcs (REAL);

theorem :: FUZZY_5:20
{f where f is Function of REAL,REAL, a,b is Real:
        for th be Real holds f.th= 1/2∗sin(a∗th+b)+1/2}
c=Membership_Funcs (REAL);

theorem :: FUZZY_5:21
for a,b be Real, f be Function of REAL,REAL st
(for th be Real holds f.th= 1/2∗sin(a∗th+b)+1/2)
holds f is FuzzySet of REAL;

theorem :: FUZZY_5:22
{f where f is Function of REAL,REAL :
ex a,b be Real st for th be Real holds f.th= 1/2∗cos(a∗th+b)+1/2}
 c= Membership_Funcs (REAL);

theorem :: FUZZY_5:23
for a,b be Real, f be Function of REAL,REAL st
(for th be Real holds f.th= 1/2∗cos(a∗th+b)+1/2)
holds f is FuzzySet of REAL;

theorem :: FUZZY_5:24
for a,b be Real, f be FuzzySet of REAL st
(a<>0 & for th be Real holds f.th= 1/2∗sin(a∗th+b)+1/2)
holds
f is normalized;
```

theorem *:: FUZZY_5:25:: TrF10:*
for f **be** FuzzySet **of** REAL **st**
f in {f **where** f **is** Function **of** REAL,REAL :
ex a,b **be** Real **st** a<>0 **&** **for** th **be** Real **holds** f.th= 1/2*sin(a*th+b)+1/2}
holds
f **is** normalized;

theorem *:: FUZZY_5:26*
for f **be** FuzzySet **of** REAL
for a,b **be** Real **st** a<>0 **&** **for** th **be** Real **holds** f.th= 1/2*cos(a*th+b)+1/2
holds
f **is** normalized;

theorem *:: FUZZY_5:27*
for f **be** FuzzySet **of** REAL **st**
f in {f **where** f **is** Function **of** REAL,REAL :
ex a,b **be** Real **st** a<>0 **&** **for** th **be** Real **holds** f.th= 1/2*cos(a*th+b)+1/2}
holds
f **is** normalized;

:::: periodicity of membership functions

theorem *:: FUZZY_5:28*
for F **being** Function **of** REAL,REAL, a,b,c,d **being** Real, i **be** Integer **st**
a<>0 **&** i<>0 **&** **for** x **be** Real **holds** F.x= max(0,min(1, c*sin(a*x+b)+d))
holds
F **is** (2 * PI)/a * i −periodic;

theorem *:: FUZZY_5:29*
for F **being** Function **of** REAL,REAL, a,b,c,d **being** Real **st**
for x **be** Real **holds** F.x= max(0,min(1, c*sin(a*x+b)+d))
holds
F **is** periodic;

theorem *:: FUZZY_5:30*
{f **where** f **is** Function **of** REAL,REAL :
ex a,b **be** Real **st** **for** th **be** Real **holds** f.th= max(0, sin(a*th+b))}
c= Membership_Funcs (REAL);

theorem *:: FUZZY_5:31*
for a,b **be** Real, f **be** Function **of** REAL,REAL **st**
(**for** x **be** Real **holds** f.x= max(0, sin(a*x+b)))
holds f **is** FuzzySet **of** REAL;

theorem *:: FUZZY_5:32*
{f **where** f **is** Function **of** REAL,REAL :
ex a,b **be** Real **st** **for** x **be** Real **holds** f.x= max(0, cos(a*x+b))}
c= Membership_Funcs (REAL);

theorem *:: FUZZY_5:33*
for a,b **be** Real, f **be** Function **of** REAL,REAL **st**
(**for** x **be** Real **holds** f.x = max(0, cos(a*x+b)))
holds f **is** FuzzySet **of** REAL;

theorem *:: FUZZY_5:34*
{f **where** f **is** Function **of** REAL,REAL, a,b,c,d **is** Real:
for x **be** Real **holds** f.x= max(0,min(1, c*sin(a*x+b)+d))}
c= {f **where** f,g **is** Function **of** REAL,REAL :
for x **be** Real **holds** f.x= max(0,min(1, g.x))};

theorem *:: FUZZY_5:35*
{f **where** f **is** Function **of** REAL,REAL, a,b,c,d **is** Real:
for x **be** Real **holds** f.x= max(0,min(1, c*sin(a*x+b)+d))}
c= Membership_Funcs (REAL);

theorem *:: FUZZY_5:36*
for f **being** Function **of** REAL,REAL, a,b,c,d **being** Real **st**

```
for x be Real holds f.x= max(0,min(1, c*sin(a*x+b)+d))
holds f is FuzzySet of REAL;

theorem :: FUZZY_5:37
{f where f is Function of REAL,REAL, a,b,c,d is Real:
 for x be Real holds f.x= max(0,min(1, c*arctan(a*x+b)+d))}
c= {f where f,g is Function of REAL,REAL :
     for x be Real holds f.x= max(0,min(1, g.x))};

theorem :: FUZZY_5:38
{f where f is Function of REAL,REAL, a,b,c,d is Real:
for x be Real holds f.x= max(0,min(1, c*arctan(a*x+b)+d))}
c= Membership_Funcs (REAL);

theorem :: FUZZY_5:39
for f being Function of REAL,REAL, a,b,c,d being Real st
for x be Real holds f.x= max(0,min(1, c*arctan(a*x+b)+d))
holds f is FuzzySet of REAL;

theorem :: FUZZY_5:40
for x being Real holds
 |. sin x .| <= |.x.|;

theorem :: FUZZY_5:41
for x,y being Real holds |. sin x - sin y .| <= |.x-y.|;

theorem :: FUZZY_5:42
for f be Function of REAL,REAL, a,b be Real st
for th be Real holds f.th = 1/2*sin(a*th+b)+1/2
holds f is continuous;

theorem :: FUZZY_5:43
for f be Function of REAL,REAL, a,b be Real st
for x be Real holds f.x = 1/2*sin(a*x+b)+1/2
holds f is continuous;

theorem :: FUZZY_5:44
for f be Function of REAL,REAL, a,b,c,d,r,s be Real st
for x be Real holds f.x= max(r,min(s, c*sin(a*x+b)+d))
holds f is Lipschitzian;

theorem :: FUZZY_5:45
for f be Function of REAL,REAL, a,b,c,d be Real st
for x be Real holds f.x= max(0,min(1, c*sin(a*x+b)+d))
holds f is Lipschitzian;


:::: membership functions from Gaussian function

theorem :: FUZZY_5:46
for a,b be Real, f be Function of REAL,REAL st
(b <> 0 & for x be Real holds f.x= exp_R(-(x-a)^2/(2*b^2)))
holds
f is FuzzySet of REAL;

theorem :: FUZZY_5:47
for a,b be Real, f be Function of REAL,REAL st
(b <> 0 & for x be Real holds f.x= exp(-(x-a)^2/(2*b^2)))
holds
f is FuzzySet of REAL;

theorem :: FUZZY_5:48
for a,b be Real st b<>0 holds
{f where f is Function of REAL,REAL :
for x be Real holds f.x= exp(-(x-a)^2/(2*b^2))}
 c= Membership_Funcs (REAL);

theorem :: FUZZY_5:49
for a,b be Real, f be FuzzySet of REAL st
```

```
(for x be Real holds f.x= exp(-(x-a)^2/(2*b^2)))
holds
f is normalized;

theorem :: FUZZY_5:50
for x be Real st exp_R x = 1 holds x = 0;

theorem :: FUZZY_5:51
for x be Real st exp x = 1 holds x = 0;

theorem :: FUZZY_5:52:: GauF02:
for a,b be Real, f be FuzzySet of REAL st
(b<>0 & for x be Real holds f.x= exp(-(x-a)^2/(2*b^2)))
holds
f is strictly-normalized;

theorem :: FUZZY_5:53
for a,b be Real, f be Function of REAL,REAL st
(b<>0 & for x be Real holds f.x= exp_R(-(x-a)^2/(2*b^2)))
holds
f is continuous;

theorem :: FUZZY_5:54
for a,b,c,r,s be Real, f be Function of REAL,REAL st
( b<>0 & for x be Real holds f.x= max(r,min(s, exp_R(-(x-a)^2/(2*b^2))+c)) )
holds
f is continuous;

theorem :: FUZZY_5:55
for a,b,c be Real, f be Function of REAL,REAL st
( b<>0 & for x be Real holds f.x= max(0,min(1, exp_R(-(x-a)^2/(2*b^2))+c)) )
holds
f is continuous;

theorem :: FUZZY_5:56
for a,b,c be Real, f be Function of REAL,REAL st
( b<>0 & for x be Real holds f.x= max(0,min(1, exp_R(-(x-a)^2/(2*b^2))+c)) )
holds
f is FuzzySet of REAL;

theorem :: FUZZY_5:57
{f where f is Function of REAL,REAL, a,b,c is Real:
 b <> 0 & for x be Real holds f.x= max(0,min(1, exp_R(-(x-a)^2/(2*b^2))+c))}
 c= Membership_Funcs (REAL);


:::: S or Z type Membership function

theorem :: FUZZY_5:58
for f be Function of REAL,REAL, a,b,r,s be Real st
for x be Real holds f.x= max(r,min(s, AffineMap (a,b).x))
holds f is Lipschitzian;

theorem :: FUZZY_5:59
for f be Function of REAL,REAL, a,b be Real st
for x be Real holds f.x= max(0,min(1, AffineMap (a,b).x))
holds f is Lipschitzian;

theorem :: FUZZY_5:60:: MM70:
for f be Function of REAL,REAL, a,b be Real st
for x be Real holds f.x= max(0,min(1, AffineMap (a,b).x))
holds f is FuzzySet of REAL;


theorem :: FUZZY_5:61
{f where f is Function of REAL,REAL, a,b is Real:
 for x be Real holds f.x= max(0,min(1, AffineMap (a,b).x))}
 c= Membership_Funcs (REAL);
```

:::: symmetrical Triangular or Trapezoidal Fuzzy Sets

theorem *:: FUZZY_5:62*
for a,b **be** Real, f **be** Function **of** REAL,REAL **st**
(**for** x **be** Real **holds** f.x = max(0,1−|.(x−a)/b.|))
holds f **is** FuzzySet **of** REAL;

theorem *:: FUZZY_5:63*
for a,b **be** Real **st** b > 0 **holds**
for x **be** Real **holds**
TriangularFS (a−b,a,a+b).x = max(0,1−|.(x−a)/b.|);

theorem *:: FUZZY_5:64*
for a,b **be** Real, f **be** FuzzySet **of** REAL **st**
b > 0 **&** (**for** x **be** Real **holds** f.x = max(0,1−|.(x−a)/b.|))
holds f **is** triangular;

theorem *:: FUZZY_5:65:: TR8:*
for a,b **be** Real, f **be** FuzzySet **of** REAL **st**
b > 0 **&** (**for** x **be** Real **holds** f.x = max(0,1−|.(x−a)/b.|))
holds f **is** strictly−normalized;

theorem *:: FUZZY_5:66*
for f **be** Function **of** REAL,REAL, a,b,c **be** Real **st**
(b <> 0 **&** **for** x **be** Real **holds** f.x= max(0,min(1, c∗(1−|.(x−a)/b.|))))
holds f **is** FuzzySet **of** REAL;

theorem *:: FUZZY_5:67*
for f **be** Function **of** REAL,REAL, a,b **be** Real **st**
b>0 **&** **for** x **be** Real **holds** f.x = max(0,1−|.(x−a)/b.|)
holds f **is** continuous;

theorem *:: FUZZY_5:68*
for f **be** Function **of** REAL,REAL, a,b,c,r,s **be** Real **st**
(b <> 0 **&** **for** x **be** Real **holds** f.x= max(r,min(s, c∗(1−|.(x−a)/b.|))))
holds f **is** Lipschitzian;

theorem *:: FUZZY_5:69*
for f **be** Function **of** REAL,REAL, a,b,c **be** Real **st**
(b <> 0 **&** **for** x **be** Real **holds** f.x= max(0,min(1, c∗(1−|.(x−a)/b.|))))
holds f **is** Lipschitzian;

theorem *:: FUZZY_5:70*
{f **where** f **is** Function **of** REAL,REAL, a,b **is** Real:
b > 0 **&** **for** x **be** Real **holds** f.x = max(0,1−|.(x−a)/b.|)}
c= Membership_Funcs (REAL);

theorem *:: FUZZY_5:71*
{f **where** f **is** Function **of** REAL,REAL, a,b,c,d **is** Real:
b <> 0 **&** **for** x **be** Real **holds** f.x= max(0,min(1, c∗(1−|.(x−a)/b.|)))}
c= Membership_Funcs (REAL);

:::: asymmetry Trapezoidal or Triangular membership function

theorem *:: FUZZY_5:72*
for a,b,p,q,s **be** Real **holds**
(AffineMap (a,b)|].−infty,s.[ ) +∗ (AffineMap (p,q)|[.s,+infty.[ )
is Function **of** REAL,REAL;

theorem *:: FUZZY_5:73:: asymTT1:*
for a,b,p,q **be** Real, f **be** Function **of** REAL,REAL **st**
for x **be** Real **holds**
f.x = max(0,min(1, (((AffineMap (a,b))|(].−infty,(q−b)/(a−p).[)) +∗
((AffineMap (p,q))|([.(q−b)/(a−p),+infty.[))) .x))
holds f **is** FuzzySet **of** REAL;

theorem *:: FUZZY_5:74*

```
for a,b,p,q be Real st
a > 0 & p > 0 & (−b)/a < q/p
holds (−b)/a < (q−b)/(a+p) & (q−b)/(a+p) < q/p & (a*q+b*p)/(a+p) > 0;

theorem :: FUZZY_5:75
for a,b,c be Real st a<b & b<c holds
TriangularFS (a,b,c). a = 0 &
TriangularFS (a,b,c). b = 1 &
TriangularFS (a,b,c). c = 0;

theorem :: FUZZY_5:76
for a,b,c,d be Real st a < b & b < c & c < d holds
TrapezoidalFS (a,b,c,d). a = 0 &
TrapezoidalFS (a,b,c,d). b = 1 &
TrapezoidalFS (a,b,c,d). c = 1 &
TrapezoidalFS (a,b,c,d). d = 0;

theorem :: FUZZY_5:77
for a,b,p,q,s be Real st
a > 0 & p > 0 & (s−b)/a = (s−q)/(−p)
holds (s−b)/a= (q−b)/(a+p) & (s−q)/(−p) = (q−b)/(a+p);

theorem :: FUZZY_5:78
for a,b,p,q be Real st
a > 0 & p > 0 & (−b)/a < q/p & (1−b)/a = (1−q)/(−p) holds
 for x be Real holds
(TriangularFS ((−b)/a,(1−b)/a,q/p)).x
= max(0,min(1, ((AffineMap (a,b) |].−infty,(q−b)/(a+p).[) +*
                  (AffineMap (−p,q)|[.(q−b)/(a+p),+infty.[)).x ));

theorem :: FUZZY_5:79
for a,b,p,q,s be Real st
a > 0 & p > 0 & (s−b)/a < (s−q)/(−p) holds
(s−b)/a < (q−b)/(a+p) & (q−b)/(a+p) < (s−q)/(−p) ;

theorem :: FUZZY_5:80
for a,b,p,q be Real st
a > 0 & p > 0 & (1−b)/a < (1−q)/(−p) holds
 for x be Real holds
(TrapezoidalFS ((−b)/a,(1−b)/a,(1−q)/(−p),q/p)).x
= max(0,min(1, ( (AffineMap (a,b) |].−infty,(q−b)/(a+p).[) +*
                  (AffineMap (−p,q)|[.(q−b)/(a+p),+infty.[) ) .x ));

theorem :: FUZZY_5:81
for x,y be Real st x >= 0 & x <= y
holds |.x.| <= |.y.|;

theorem :: FUZZY_5:82
for x,y be Real st x <= 0 & y <= x
holds |.x.| <= |.y.|;

theorem :: FUZZY_5:83
for a,b,p,q be Real, f be Function of REAL,REAL st a > 0 & p > 0 &
f = (AffineMap (a,b)|].−infty,(q−b)/(a+p).[) +*
    (AffineMap (−p,q)|[.(q−b)/(a+p),+infty.[)
holds f is Lipschitzian;

theorem :: FUZZY_5:84
for a,b,p,q be Real st a > 0 & p > 0
holds
ex r being Real st
 ( 0 < r &
 ( for x1, x2 being Real st
x1 in dom ((AffineMap (a,b)|].−infty,(q−b)/(a+p).[)
          +* (AffineMap (−p,q)|[.(q−b)/(a+p),+infty.[)) &
x2 in dom ((AffineMap (a,b)|].−infty,(q−b)/(a+p).[)
          +* (AffineMap (−p,q)|[.(q−b)/(a+p),+infty.[)) holds
  |. ((AffineMap (a,b)|].−infty,(q−b)/(a+p).[)
```

```
          +* (AffineMap (-p,q)|[.(q-b)/(a+p),+infty.[]) . x1
  - ((AffineMap (a,b)|].-infty,(q-b)/(a+p).[)
          +* (AffineMap (-p,q)|[.(q-b)/(a+p),+infty.[]) . x2.|
  <= r * |.(x1 - x2).| ) );
```

theorem *:: FUZZY_5:85*
for a,b,p,q,r,s **be** Real, f **be** Function **of** REAL,REAL **st**
a > 0 **&** p > 0 **&**
for x **be** Real **holds**
f.x = max(r,min(s, ((AffineMap (a,b)|].−infty,(q−b)/(a+p).[) +*
(AffineMap (−p,q)|[.(q−b)/(a+p),+infty.[)) .x))
holds f **is** Lipschitzian;

theorem *:: FUZZY_5:86*
for a, b, c **being** Real **st** a < b **&** b < c **holds**
for x **being** Real **holds**
TriangularFS (a,b,c).x =
max(0,min(1, ((AffineMap (1/(b−a),− a/(b−a))|].−infty,b.[) +*
(AffineMap (− 1/(c−b),c/(c−b))|[.b,+infty.[)).x));

theorem *:: FUZZY_5:87*
for a, b, c, d **being** Real **st** a < b **&** b < c **&** c < d **holds**
for x **being** Real **holds**
TrapezoidalFS (a,b,c,d).x =
max(0,min(1, (
(AffineMap (1/(b−a),− a/(b−a))|].−infty,(b*d−a*c)/(d−c+b−a).[) +*
(AffineMap (− 1/(d−c),d/(d−c))|[.(b*d−a*c)/(d−c+b−a),+infty.[)).x));

theorem *:: FUZZY_5:88*
for a,b,p,q **be** Real, f **be** Function **of** REAL,REAL **st**
a > 0 **&** p > 0 **&**
for x **be** Real **holds**
f.x = max(0,min(1, (((AffineMap (a,b))|(].−infty,(q−b)/(a+p).[)) +*
((AffineMap (−p,q))|([.(q−b)/(a+p),+infty.[))) .x))
holds f **is** Lipschitzian;

theorem *:: FUZZY_5:89*
for a, b, c **being** Real **st** a < b **&** b < c **holds**
TriangularFS (a,b,c) **is** Lipschitzian;

theorem *:: FUZZY_5:90*
for a, b, c, d **being** Real **st** a < b **&** b < c **&** c < d **holds**
TrapezoidalFS (a,b,c,d) **is** Lipschitzian;

theorem *:: FUZZY_5:91*
for a,b,p,q **be** Real, f **be** FuzzySet **of** REAL **st**
a > 0 **&** p > 0 **&** (−b)/a < q/p **&** (1−b)/a = (1−q)/(−p) **&**
for x **be** Real **holds**
f.x
= max(0,min(1, ((AffineMap (a,b) |].−infty,(q−b)/(a+p).[) +*
(AffineMap (−p,q)|[.(q−b)/(a+p),+infty.[)) .x))
holds f **is** triangular **&** f **is** strictly−normalized;

theorem *:: FUZZY_5:92*
for a,b,p,q **be** Real, f **be** FuzzySet **of** REAL **st**
a > 0 **&** p > 0 **&** (1−b)/a < (1−q)/(−p) **&**
for x **be** Real **holds**
f.x
= max(0,min(1, ((AffineMap (a,b) |].−infty,(q−b)/(a+p).[) +*
(AffineMap (−p,q)|[.(q−b)/(a+p),+infty.[)) .x))
holds f **is** trapezoidal **&** f **is** normalized;

theorem *:: FUZZY_5:93*
{f **where** f **is** FuzzySet **of** REAL:f **is** triangular}
c= Membership_Funcs (REAL);

theorem *:: FUZZY_5:94*
{TriangularFS (a,b,c) **where** a,b,c **is** Real : a < b **&** b < c}

c= Membership_Funcs (REAL);

theorem *:: FUZZY_5:95*
{f **where** f **is** FuzzySet **of** REAL:f **is** trapezoidal}
c= Membership_Funcs (REAL);

theorem *:: FUZZY_5:96*
{TrapezoidalFS (a,b,c,d) **where** a,b,c,d **is** Real : a < b **&** b < c **&** c < d}
c= Membership_Funcs (REAL);